A ZERO-STABLE ALGORITHM FOR FOURTH ORDER INITIAL AND BOUNDARY VALUE PROBLEMS INTEGRATION

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Abstract: The paper focuses on derivation of fifth order hybrid linear multistep block method (HLMBM) for the solution of fourth order initial value problems (IVPs) in ordinary differential equations (ODEs). We demonstrate the possibility of direct integration of fourth order boundary value problems using the HLMBM. Collocation technique is adopted in the derivation of the HLMBM which is applied as simultaneous integrator to fourth order initial and boundary value problems. The HLMBM possesses the desirable feature of being self-starting as the implementation is in block form. Numerical examples are included to demonstrate the validity and applicability of the proposed OLMBM and comparisons are made with the exact solution to show the desirability of the method.
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Introduction

The general m[th]-order initial and boundary value problems are of the form

$$y^{m}(x)=f(x,y,y',...,y^{m-1})$$

Subject to either initial conditions or boundary conditions.
In this paper, we focus specifically on case m = 4, that is the fourth order differential equation

$$y^{iv}(x)=f(x,y,y',y'',y''') \qquad (1)$$

Subject to the initial conditions

$$y(x_0)=a_0, y'(x_0)=b_0, y''(x_0)=d_0, y'''(x_0)=e_0$$

or the boundary conditions

$$y(x_0)=a_0, y'(x_0)=b_0, y(x_N)=d_N, y(x_N)=e_N$$

and formulate one-step self-starting continuous hybrid linear multistep method of the form

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h^4\left[\sum_{i=0}^{k}\beta_i f_{n+i}+\sum_{i=1}^{k}\beta_i f_{n+w}\right]$$

Where: $\alpha_k=1, \beta_k\neq 0$ and k = 1 is the step number.

Equation (1) has many applications in engineering, science and management such as the bending of an elastic beam which is described with a fourth-order boundary value problem, the reaction and diffusion of chemicals, the dynamics of populations in biology, the development and treatment of diseases in medicine, molecular dynamics, the motion of rocket and several other areas. So, the demand for the solution of differential equations
(DEs) is on the increase as the quest for numerical methods has increasingly been of much interest to researchers owing to the fact that most of these DEs are difficult to solve or their analytical solutions do not exist.
The approach of reducing (1) to a system of first order differential equations and then applying the various methods available for solving systems of first order Initial Value Problems (IVPs) has been extensively discussed in the literature (Adesanya *et al.*, 2012; Butcher, 2008; Lambert, 1991; Henrici, 1962; Hairer *et al.*, 1993; Dormand, 1996). This approach had been reported to increase the number of equations four times and thereby more function evaluations need to be evaluated as this result to a longer execution time and more computational effort (Jator, 2008; Awoyemi *et al.*, 2011; Waeleh *et al.*, 2012; Mehrkanoon, 2011). Moreover, Bun and Vasil'yer (1992) reported that the system of equations to be solved when the method of reduction is applied cannot be solved explicitly with respect to the derivatives of the highest order.
Awoyemi (2003, 2005), Kayode (2008a, 2008b) succeeded in applying numerical algorithm to directly solve a general fourth order initial value problems of the form (1). However, all these methods were implemented in predictor-corrector mode and hence, according to Jator (2008), the implementation of such schemes is more costly since the subroutines for incorporating the starting values lead to lengthy computational time. Besides, they advance the numerical integration of the ordinary differential equations in one-step at a time, which leads to overlapping of the piecewise polynomials solution model (Yusuph, 2004). To address the setback of the predictor-corrector method; Vigo-Aguiar and Ramos (2006), Jator (2007), Yap and Ismail (2015), Costabile and Napoli (2015), Hussain *et al*. (2016), among others independently proposed block method for solving higher order ordinary differential equation which does not require the development of separate predictors but simultaneously generate approximation at different grid points within the interval of integration without overlapping as experienced in the predictor-corrector method.
The aim of developing new methods has always been to improve on the efficiency and convergence of existing methods with the ultimate aim of reducing the error of approximation.
Thus, in what immediately follows, we shall formulate one-step method to directly integrate fourth order initial value problem and subsequently, the implementation shall be extended to boundary value problems as well.

Derivation of HLMBM

We assume an approximate solution of the form

$$y(x)=\sum_{r=0}^{n}a_r x^r, \qquad n=s+k-1 \qquad (2)$$

Where $x\in[x_i, x_{i+j}]$, the number of interpolation points used s ≥ 4 and k $\geq$ s where k is the number of collocation points.

Interpolating and collocating at $x = x_{n+s}$, $s = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ and $x = x_{n+k}$, $k = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$ respectively gives a system of equations needed to be used in determining the unique values of a_r, $r = 0(1)8$ written as

AX = R (3)

where

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+\frac{1}{4}} & x_{n+\frac{1}{4}}^2 & x_{n+\frac{1}{4}}^3 & x_{n+\frac{1}{4}}^4 & x_{n+\frac{1}{4}}^5 & x_{n+\frac{1}{4}}^6 & x_{n+\frac{1}{4}}^7 & x_{n+\frac{1}{4}}^8 \\ 1 & x_{n+\frac{1}{3}} & x_{n+\frac{1}{3}}^2 & x_{n+\frac{1}{3}}^3 & x_{n+\frac{1}{3}}^4 & x_{n+\frac{1}{3}}^5 & x_{n+\frac{1}{3}}^6 & x_{n+\frac{1}{3}}^7 & x_{n+\frac{1}{3}}^8 \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 & x_{n+\frac{1}{2}}^7 & x_{n+\frac{1}{2}}^8 \\ 0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & 840x_n^3 & 1680x_n^4 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{1}{4}} & 360x_{n+\frac{1}{4}}^2 & 840x_{n+\frac{1}{4}}^3 & 1680x_{n+\frac{1}{4}}^4 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{1}{3}} & 360x_{n+\frac{1}{3}}^2 & 840x_{n+\frac{1}{3}}^3 & 1680x_{n+\frac{1}{3}}^4 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{1}{2}} & 360x_{n+\frac{1}{3}}^2 & 840x_{n+\frac{1}{2}}^3 & 1680x_{n+\frac{1}{2}}^4 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+1} & 360x_{n+1}^2 & 840x_{n+1}^3 & 1680x_{n+1}^4 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_n & 2520x_n^2 & 6720x_n^3 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{1}{4}} & 2520x_{n+\frac{1}{4}}^2 & 6720x_{n+\frac{1}{4}}^3 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{1}{3}} & 2520x_{n+\frac{1}{3}}^2 & 6720x_{n+\frac{1}{3}}^3 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+\frac{1}{2}} & 2520x_{n+\frac{1}{2}}^2 & 6720x_{n+\frac{1}{2}}^3 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+1} & 2520x_{n+1}^2 & 6720x_{n+1}^3 \end{bmatrix},$$

$X = [a_0 \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7 \ a_8]^T$, and

$$R = \left[y_n \ y_{n+\frac{1}{4}} \ y_{n+\frac{1}{3}} \ y_{n+\frac{1}{2}} \ f_n \ f_{n+\frac{1}{4}} \ f_{n+\frac{1}{3}} \ f_{n+\frac{1}{2}}, f_{n+1} \right]^T$$

Solving for a_r's, $r = 0(1)8$ in (3) and substituting the resulting equations in (2), the continuous equation is obtained as

$$y(x) = \sum \alpha_s y_{n+s} + h^4 (\sum \beta_k f_{n+k}), \ s = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \ k = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1 \qquad (4)$$

where α_s and β_k are parameters to be determined.

Evaluating (4) at $x = x_{n+1}$, we have

$$y_{n+1} = 64 y_{n+\frac{1}{4}} - 6y_n - 81 y_{n+\frac{1}{3}} + 24 y_{n+\frac{1}{2}} + h^4 \Big(\frac{503}{4976640} f_{n+1} + \frac{499}{69120} f_{n+\frac{1}{2}} - \frac{137}{61440} f_{n+\frac{1}{3}} + \frac{427}{77760} f_{n+\frac{1}{4}} - \frac{137}{829440} f_n \Big) \qquad (5)$$

Evaluating the first, second and third derivatives of (4) at $x = x_{n+k}, k = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$ and solving the resulting equations and (5) simultaneously gives a discrete block formula in the form

	y_n	y_n'	y_n''	y_n'''	f_n	$f_{n+\frac{1}{4}}$	$f_{n+\frac{1}{3}}$	$f_{n+\frac{1}{2}}$	f_{n+1}
$y_{n+\frac{1}{4}}$	1	$\frac{1}{4}$	$\frac{1}{32}$	$\frac{1}{384}$	$\frac{4199}{41287680}$	$\frac{47}{1474560}$	$\frac{-1611}{9175040}$	$\frac{53}{258048}$	$\frac{-17}{27525120}$
$y_{n+\frac{1}{3}}$	1	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{1}{162}$	$\frac{257}{918540}$	$\frac{496}{688905}$	$\frac{-1}{1701}$	$\frac{8}{76545}$	$\frac{-11}{5511240}$
$y_{n+\frac{1}{2}}$	1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{48}$	$\frac{179}{161280}$	$\frac{19}{5040}$	$\frac{-99}{35840}$	$\frac{1}{2016}$	$\frac{-1}{107520}$
y_{n+1}	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{13}{1260}$	$\frac{16}{315}$	$\frac{-9}{280}$	$\frac{4}{315}$	0
$y'_{n+\frac{1}{4}}$		1	$\frac{1}{4}$	$\frac{1}{32}$	$\frac{2497}{1720320}$	$\frac{59}{16128}$	$\frac{-3483}{1146880}$	$\frac{233}{430080}$	$\frac{-107}{10321920}$
$y'_{n+\frac{1}{3}}$		1	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{11}{3780}$	$\frac{704}{76545}$	$\frac{-65}{9072}$	$\frac{32}{25515}$	$\frac{-29}{1224720}$
$y'_{n+\frac{1}{2}}$		1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{119}{26880}$	$\frac{37}{1260}$	$\frac{-351}{17920}$	$\frac{5}{1344}$	$\frac{-11}{161280}$
y'_{n+1}		1	1	$\frac{1}{2}$	$\frac{13}{420}$	$\frac{64}{315}$	$\frac{-81}{560}$	$\frac{8}{105}$	$\frac{-1}{1008}$
$y''_{n+\frac{1}{4}}$			1	$\frac{1}{4}$	$\frac{147}{10240}$	$\frac{7}{144}$	$\frac{-783}{20480}$	$\frac{38}{3645}$	$\frac{-23}{184320}$
$y''_{n+\frac{1}{3}}$			1	$\frac{1}{3}$	$\frac{301}{14580}$	$\frac{928}{10935}$	$\frac{-13}{216}$	$\frac{38}{3645}$	$\frac{-17}{87480}$
$y''_{n+\frac{1}{2}}$			1	$\frac{1}{2}$	$\frac{1}{30}$	$\frac{7}{45}$	$\frac{-27}{320}$	$\frac{1}{48}$	$\frac{-1}{2880}$
y''_{n+1}			1	1	$\frac{1}{20}$	$\frac{32}{45}$	$\frac{-27}{40}$	$\frac{2}{5}$	$\frac{1}{72}$
$y'''_{n+\frac{1}{4}}$				1	$\frac{581}{7680}$	$\frac{49}{120}$	$\frac{-1431}{5120}$	$\frac{89}{1920}$	$\frac{-13}{7680}$
$y'''_{n+\frac{1}{3}}$				1	$\frac{61}{810}$	$\frac{544}{1215}$	$\frac{-7}{30}$	$\frac{2}{45}$	$\frac{-1}{1215}$
$y'''_{n+\frac{1}{2}}$				1	$\frac{37}{480}$	$\frac{2}{5}$	$\frac{-27}{320}$	$\frac{13}{120}$	$\frac{-9}{960}$
y'''_{n+1}				1	$\frac{-1}{30}$	$\frac{32}{15}$	$\frac{-27}{10}$	$\frac{22}{15}$	$\frac{2}{15}$

Analysis of the basic properties of the Method

Order of the HLMBM

Following Henrici (1962), the approach adopted in Fatunla (1991, 1994) and Lambert (1973), we define the local truncation error associated with equation (6) by the difference operator

$$L[y(x):h] = \sum_{j=0}^{k} \left[\alpha_j y(x_n + jh) - h^4 \beta_j f(x_n + jh) \right] \quad (7)$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding (7) in Taylor series about the point x, we obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_{p+4} h^{p+4} y^{p+4}(x)$$

where the C_0 , C_1 , C_2 ,C_p ,... are obtained as

$$C_0 = \sum_{j=0}^{k} \alpha_j \text{ , } C_1 = \sum_{j=1}^{k} j\alpha_j \text{ , } C_2 = \frac{1}{2!}\sum_{j=1}^{k} j^2 \alpha_j \text{ , } C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2)\sum_{j=1}^{k} \beta_j j^{q-3}\right]$$

In the spirit of Lambert (1973), equations (5) and (6) are of order p if

$C_0 = C_1 = C_2 = \ldots C_p = \ldots C_{p+3} = 0$ and $C_{p+4} \neq 0$

The $C_{p+4} \neq 0$ is called the error constant and $C_{p+4} h^{p+4} y^{p+4}(x_n)$ is the principal local truncation error at the point x_n .

According to the definition above, equations (5) and (6) are all of order 5 with the error constants $C_{p+4} = \left[\frac{-275}{5159780352}\right]^T$

and

$$C_{p+2} = \begin{bmatrix} \frac{41}{33973862400} & \frac{551}{142851340800} & \frac{11}{619315200} & \frac{1}{7257600} & \frac{53}{2642411520} & \frac{1}{22044960} & \frac{1}{7741440} \\ \frac{-1}{14515200} & \frac{59}{247726080} & \frac{13}{35271936} & \frac{1}{1548288} & \frac{-1}{241920} & \frac{7}{4423680} & \frac{13}{8398080} & \frac{1}{552960} & \frac{-1}{34560} \end{bmatrix}^T$$

respectively.

With the order p = 5 > 1, we establish the consistency of the method (Jator, 2008;Henrici, 1962).

The first characteristic polynomial of the hybrid block method (6) is given by

$$\rho(R) = \det(RA^0 - A^1) \qquad (8)$$

where A^0 is 16 by 16 identity matrix and

$$A^1 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{32} & 0 & 0 & 0 & \frac{1}{384} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{18} & 0 & 0 & 0 & \frac{1}{162} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{48} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{32} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{18} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Substituting A^0 and A^1 in (8), we obtain $\rho(R) = R^{12}(R^4 - 1)$.

According to Fatunla (1988, 1991), the method is zero-stable since $\rho(R)=0$ satisfies $|R_j|\le 1$, $j=1$ and for those roots with $|R_j|=1$, the multiplicity does not exceed four.

Region of absolute stability of HLMBM
For the region of absolute stability, the following definitions are considered.
Given the stability polynomial

$$\pi(z,\bar{h})=\rho(z)-\bar{h}\sigma(z)=0 \qquad (9)$$

where $\bar{h}=h^2\lambda^2$ and $\lambda=\dfrac{df}{dy}$ are assumed constants.

The scheme (6) is said to be absolutely stable if for a given $\bar{h}$ all the roots Z_s of (9) satisfy $|z_s|<1$, s=1,2,…n, where $\bar{h}=\lambda h$

Definition 1.1: The region $\Re$ of the complex $\bar{h}$-plane such that the roots of $\pi(z,\bar{h})=0$ lies within the unit circle whenever $\bar{h}$ lies in the interior of the region is called the region of absolute stability.
Remark: Let $\Re$ be the boundary of the region $\Re$. Since the roots of the stability polynomial are continuous functions of $\bar{h}$, $\bar{h}$ will lie on $\Re$ when one of the roots of the $\pi(z,\bar{h})=0$ lies on the boundary of the unit circle. Thus we define (9) in terms of Euler's number, $\exp i\theta$, as follows;

$$\pi(\exp(i\theta),h)=\rho(\exp(i\theta)-\bar{h}\sigma(\exp(i\theta))=0 \qquad (10)$$

So that, the locus of the boundary $\Re$ is given by $\bar{h}(\theta)=\dfrac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$

From (5), the boundary of the region of absolute stability is

$$\bar{h}(\theta)=\frac{81\cos\frac{1}{3}\theta-24\cos\frac{1}{2}\theta-64\cos\frac{1}{4}\theta-24i\sin\frac{1}{2}\theta+81i\sin\frac{1}{3}\theta-64i\sin\frac{1}{4}\theta+\cos\theta+i\sin\theta+6}{\left(\begin{array}{l}\frac{427}{77760}i\sin\frac{1}{4}\theta-\frac{137}{61440}i\sin\frac{1}{3}\theta+\frac{449}{69120}i\sin\frac{1}{2}\theta+\frac{503}{4976640}i\sin\theta-\frac{137}{829440}\\+\frac{427}{77760}\cos\frac{1}{4}\theta-\frac{137}{61440}\cos\frac{1}{3}\theta+\frac{449}{69120}\cos\frac{1}{2}\theta+\frac{503}{4976640}\cos\theta\end{array}\right)}$$

and the region of absolute stability (RAS) is shown in Figure 5.

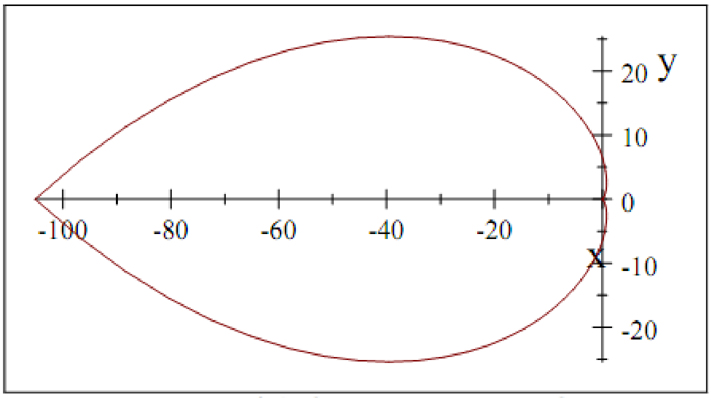

Fig. 5: RAS OF THE METHOD

Implementation of HLMBM as a block as well as a block unification method
Implementation of HLMBM
We implement the HLMBM using a written code in Mathematica 10.0 enhanced by the features NSolve[ ] for linear problems and FindRoot[ ] for nonlinear problems respectively. In what follows, we summarize how HLMBM is applied to solve initial value problems (IVPs) in a block-by-block fashion as well as applied to solve boundary value problems (BVPs) via a block unification technique.
IVPs-Block-by-block algorithm

Step 1: Choose N, $h=\dfrac{(x_N-x_0)}{N}$, on the partition Q_N.

Step 2: Solve for the values of

$$[y_r, y_s, y_v, y_1, y_r', y_s', y_v', y_1', y_r'', y_s'', y_v'', y_1'', y_r''', y_s''', y_v''', y_1''']^T$$

simultaneously on the sub-interval [x_0, x_1] as n = 0, y_0, y_0', y_0'', and y_0''' are known from the IVPs (1) and $\{r, s, v\} = \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$.

Step 3: For n = 1, generate the variables

$$[y_{r+1}, y_{s+1}, y_{v+1}, y_2, y_{r+1}', y_{s+1}', y_{v+1}', y_2', y_{r+1}'', y_{s+1}'', y_{v+1}'', y_2'', y_{r+1}''', y_{s+1}''', y_{v+1}''', y_2''']^T$$

and obtain their values by solving simultaneously on the sub-interval [x_1, x_2] as y_1, y_1', y_1'', and y_1''' are known from the previous block.

Step 4: Repeat the process for n = 2, . . . ,N - 1 to obtain the numerical solution to (1) on the sub-intervals [x_0, x_1], [x_1, x_2], . . . , [x_{N-1}, x_N].

BVPs-Block unification algorithm

Step 1: Choose N, $h = \frac{(x_N - x_0)}{N}$, on the partition Q_N.

Step 2: For n = 0, generate the variables

$$[y_r, y_s, y_v, y_1, y_r', y_s', y_v', y_1', y_r'', y_s'', y_v'', y_1'', y_r''', y_s''', y_v''', y_1''']^T$$

simultaneously on the sub-interval [x_0, x_1] and do not solve yet.

Step 3: For n = 1, generate the variables

$$[y_{r+1}, y_{s+1}, y_{v+1}, y_2, y_{r+1}', y_{s+1}', y_{v+1}', y_2', y_{r+1}'', y_{s+1}'', y_{v+1}'', y_2'', y_{r+1}''', y_{s+1}''', y_{v+1}''', y_2''']^T$$

on the sub-interval [x_1, x_2] and do not solve yet.

Step 4: Repeat the process for n = 2, . . . ,N - 1 until all the variables on the sub-intervals [x_0, x_1], [x_1, x_2], . . . , [x_{N-1}, x_N] are obtained.

Step 5: Create a single block matrix equation by the unification of all the blocks generated in Step 2 and Step 3 on Q_N.

Step 6: Solve the single block matrix equation to simultaneously obtain all the solutions for (1) on the entire [x_0, x_N].

Numerical examples

The effectiveness of the HLMBM is investigated by solving four test problems. Two IVPs and two BVPs are considered to test the performance of our method. All computations were carried out using our written Mathematica code in Mathematica 10.0.

Figures 1 – 4 show the comparison of the solution using the new numerical method HLMBM and the exact solution.

Problem 1: Consider the linear fourth order IVPs (Jator, 2008)

$$y^{iv} = y''' + y'' + y' + 2y,\ y(0) = y'(0) = y''(0) = 0,\ \ y'''(0) = 30,\ 0 \le t \le 2$$

whose theoretical solution is y(t) = $2e^{2t}$ - $5e^{-t}$ + 3cost – 9sint.

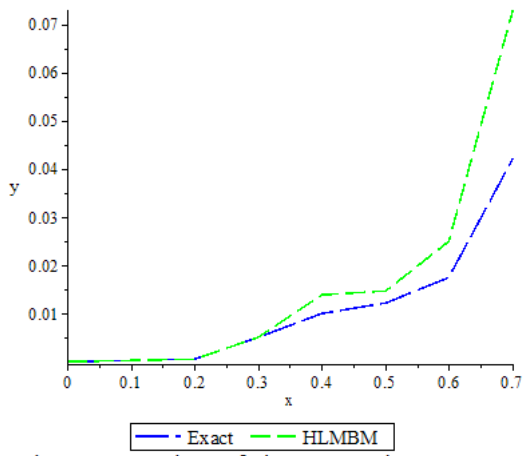

Fig. 1 :Comparison of the exact and HLMBM solutions of problem 1

Problem 2: Consider the nonlinear fourth order IVPs

$$y^{iv} = \frac{3(3 + 2\sin^2 y)\sin y}{\cos^7 y},\ y(0) = 0,\ \ y'(0) = 1,\ \ y''(0) = 0,\ \ y'''(0) = 1.$$

The problem is integrated in the interval $[0, \frac{\pi}{4}]$. The exact solution is given by y(x) = arcsin(x).

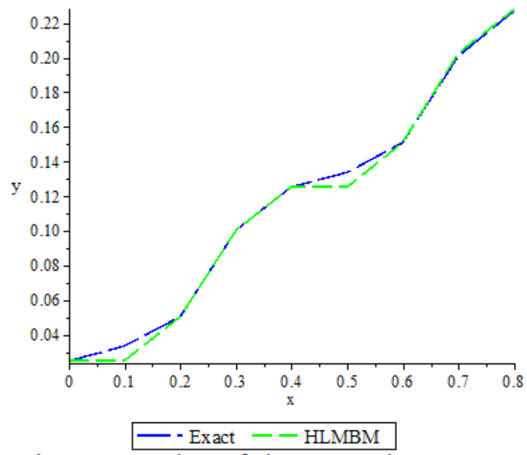

Fig. 2 :Comparison of the exact and HLMBM solutions of problem 2

Problem 3: We consider the BVPs

$$\begin{cases} y^{iv}(t) = y(t) + y'''(t) + e^t(t-3), t \in [0,1] \\ y(0) = 1,\ \ y'(0) = 0, \\ y(1) = 0,\ \ y'(1) = -e \end{cases}$$

The exact solution is given by $y(t) = (1-t)e^t$.

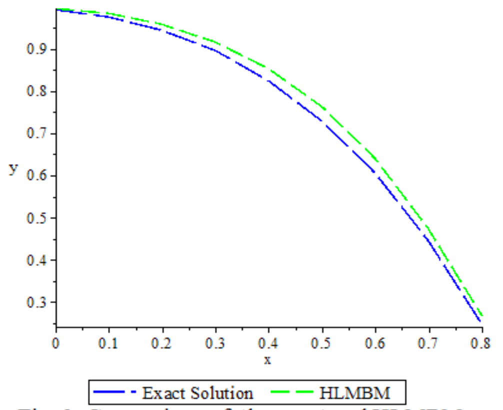

Fig. 3 :Comparison of the exact and HLMBM solutions of problem 3

Problem 4: We consider the following nonlinear BVPs

$$\begin{cases} y^{iv}(x) = y(x)^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 \\ y(0) = 0, y'(0) = 0 \\ y(1) = 1, y'(1) = 1 \end{cases}$$

The exact solution is given by $y(x) = x^5 - 2x^4 + 2x^2$
The problem is integrated in the interval [0, 1].

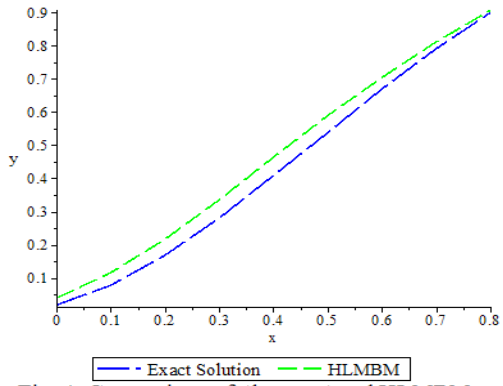

Fig. 4 :Comparison of the exact and HLMBM solutions of problem 4

Conclusion

The derivation of one-step block algorithm which is applied as simultaneous numerical integrator of fourth order initial value problems over non-overlapping intervals has been demonstrated. The method is implemented in block and extended block forms. Solutions of numerical experiments performed using HLMBM are shown in Figures 1-4 and these show that the method conveniently integrates both IVPs and BVPs of fourth order. In our future paper, we shall extend the method to solve partial differential equation through the method of lines.

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